

# Degenerations of binary Lie and nilpotent Malcev algebras <sup>1</sup>

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**Abstract.** We describe degenerations of four-dimensional binary Lie algebras, and five- and six-dimensional nilpotent Malcev algebras over  $\mathbb{C}$ . In particular, we describe all irreducible components of these varieties.

**Keywords:** Malcev algebra, binary Lie algebra, nilpotent algebra, degeneration, rigid algebra

## 1. INTRODUCTION

Degenerations of algebras is an interesting subject, which was studied in various papers (see, for example, [1–9, 12–14, 16, 17, 25, 26, 28, 30]). In particular, there are many results concerning degenerations of algebras of low dimensions from some variety defined by a set of identities. One of important problems in this direction is the description of so-called rigid algebras. These algebras are of big interest, since the closures of their orbits under the action of generalized linear group form irreducible components of a variety under consideration (with respect to Zariski topology). For example, the problem of finding rigid algebras was solved for low-dimensional associative (see [9, 25, 26]), Leibniz (see [1, 8, 28]), and Jordan (see [16, 17]) algebras. There are significantly less works where the full information about degenerations was found for some variety of algebras. This problem was solved for four-dimensional Lie algebras in [7], for nilpotent five- and six-dimensional Lie algebras in [13, 30], for two-dimensional pre-Lie algebras in [2], and for three-dimensional Novikov algebras in [3].

The notions of Malcev and binary Lie (BL for short) algebras were introduced by Malcev in [24]. The structure theory and some properties of Malcev algebras were studied by Kuzmin and other authors (see, for example, [15, 18–21, 23, 27, 29]). Note that any Lie algebra is a Malcev algebra and any Malcev algebra is a BL algebra. Note also that any alternative algebra can be turned to a Malcev algebra by defining a new multiplication  $[\cdot, \cdot]$  by  $[x, y] = xy - yx$ . Any Malcev algebra is a tangent algebra of a suitable locally analytic Moufang loop (see [21]).

In this paper we give the full information about degenerations of BL algebras of dimension 4 and nilpotent Malcev algebras of dimensions 5 and 6. More precisely, we construct a graph of primary degenerations. The vertices of this graph are isomorphism classes of algebras from the variety under consideration. An algebra  $A$  degenerates to an algebra  $B$  iff there is a path from the vertex corresponding to  $A$  to the vertex corresponding to  $B$ . Thus, we obtain a generalization of analogous results of [7, 13, 30] for Lie algebras. Also we describe rigid algebras and irreducible components for these varieties of algebras.

## 2. DEFINITIONS AND NOTATION

All spaces in this paper are considered over  $\mathbb{C}$ , and we write simply  $\dim$ ,  $\text{Hom}$  and  $\otimes$  instead of  $\dim_{\mathbb{C}}$ ,  $\text{Hom}_{\mathbb{C}}$  and  $\otimes_{\mathbb{C}}$ . An algebra  $A$  is a set with a structure of vector space and a binary operation that induces a bilinear map from  $A \times A$  to  $A$ .

Given an  $n$ -dimensional vector space  $V$ , the set  $\text{Hom}(V \otimes V, V) = V^* \otimes V^* \otimes V$  is a vector space of dimension  $n^3$ . This space has the structure of the affine variety  $\mathbb{C}^{n^3}$ . Indeed, let us fix a basis  $e_1, \dots, e_n$  of  $V$ . Then any  $\mu \in \text{Hom}(V \otimes V, V)$  is defined by structure constants  $c_{i,j}^k \in \mathbb{C}$  such that  $\mu(e_i \otimes e_j) = \sum_{k=1}^n c_{i,j}^k e_k$ . A subset of  $\text{Hom}(V \otimes V, V)$  is called closed if it can be defined by a set of polynomial equations in variables  $c_{i,j}^k$  ( $1 \leq i, j, k \leq n$ ).

Let  $T$  be a set of polynomial identities. All algebra structures on  $V$  satisfying polynomial identities from  $T$  form a Zariski-closed affine subset of the variety  $\text{Hom}(V \otimes V, V)$ . We denote this subset by  $\mathbb{L}(T)$ . The general linear group  $GL_n(\mathbb{C})$  operates on  $\mathbb{L}(T)$  by conjugation:

$$(g * \mu)(x \otimes y) = g(\mu(g^{-1}(x) \otimes g^{-1}(y)))$$

for  $x, y \in V$ ,  $\mu \in \mathbb{L}(T) \subset \text{Hom}(V \otimes V, V)$  and  $g \in GL_n(\mathbb{C})$ . Thus,  $\mathbb{L}(T)$  is decomposed into  $GL_n(\mathbb{C})$ -orbits that correspond to the isomorphism classes of algebras. Let  $O(\mu)$  denote the orbit of  $\mu \in \mathbb{L}(T)$  under the action of  $GL_n(\mathbb{C})$ . Correspondingly,  $\overline{O(\mu)}$  is the Zariski closure of  $O(\mu)$ .

Let  $A$  and  $B$  be two  $n$ -dimensional algebras satisfying identities from  $T$ . Let  $\mu$  and  $\lambda$  from  $\mathbb{L}(T)$  represent  $A$  and  $B$  respectively. We say that  $A$  degenerates to  $B$  and write  $A \rightarrow B$  if  $\lambda \in \overline{O(\mu)}$ . Note that in this case we have  $\overline{O(\lambda)} \subset \overline{O(\mu)}$ . Hence, the definition of degeneration does not depend on the choice of  $\mu$  and  $\lambda$ . We write  $A \not\rightarrow B$  if  $\lambda \notin \overline{O(\mu)}$ .

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Let  $A$  be represented by the structure  $\mu \in \mathbb{L}(T)$ . The algebra  $A$  is called *rigid* in  $\mathbb{L}(T)$  if  $O(\mu)$  is an open subset of  $\mathbb{L}(T)$ . Recall that a subset of a variety is called irreducible if it can't be represented as a union of two non-trivial closed subsets. A maximal irreducible closed subset of a variety is called an *irreducible component*. In particular,  $A$  is rigid in  $\mathbb{L}(T)$  iff  $\overline{O(\mu)}$  is an irreducible component of  $\mathbb{L}(T)$ . Let  $Irr(\mathbb{L}(T))$  and  $Rig(\mathbb{L}(T))$  denote the set of irreducible components of  $\mathbb{L}(T)$  and the set of rigid algebras in  $\mathbb{L}(T)$  respectively. It is well known that any affine variety can be represented as a finite union of its irreducible components in a unique way.

Let  $A$  be an algebra. For  $x, y, z \in A$  we define their Jacobian  $J(x, y, z)$  by the equality  $J(x, y, z) = (xy)z + (yz)x + (zx)y$ . The algebra  $A$  is called a Malcev algebra if it satisfies the identities

$$xy = -yx, \quad J(x, y, xz) = J(x, y, z)x.$$

The algebra  $A$  is called a binary Lie (BL for short) algebra if all its 2-generated subalgebras are Lie algebras. It was shown by Gainov in [11] that  $A$  is a BL algebra iff it satisfies the identities

$$xy = -yx, \quad J(x, y, xy) = 0.$$

It is easy to see that any Lie algebra is a Malcev algebra and any Malcev algebra is a BL algebra. It was shown in [10] that any three-dimensional BL algebra is a Lie algebra. The classification of four-dimensional BL algebras was obtained in [10, 22]. The classification of five-dimensional and nilpotent six-dimensional non-Lie Malcev algebras is given in [20].

Let  $BL_n$ ,  $Mal_n$  and  $Lie_n$  denote the varieties of  $n$ -dimensional BL, Malcev and Lie algebras respectively, and  $NBL_n$ ,  $NMal_n$  and  $NLie_n$  denote their subvarieties formed by nilpotent algebras.

Define the sets  $A^l$  by the equalities  $A^1 = A$  and  $A^l = A^{l-1}A$  ( $l > 1$ ). Also define the central series  $Z_l(A)$  ( $l > 0$ ) of  $A$  in the following way. We define  $Z_1(A) = Z(A)$  as the center of  $A$ , and, for  $l > 1$ ,  $Z_l(A)$  is the full inverse image of  $Z(A/Z_{l-1}(A))$  under the canonical projection from  $A$  to  $A/Z_{l-1}(A)$ .

We collect all the information that we need about the algebras under consideration in Tables 4.1, 5.1 and 6.1. In these tables in the first column we write the names of the algebras. In the second column we give the multiplication tables in some fixed basis  $e_1, \dots, e_n$  of  $V$ . All products of basis elements, which are not described in the table, are zero or can be deduced from one of the described products and the anticommutativity identity. In the third column we give the dimensions of algebras of derivations. In the columns named  $Z(A)$ ,  $A^2$  and  $A^3$  we give the dimensions of the corresponding spaces. In the column named  $Z_l(A)$  we give the dimensions of the members of central series of  $A$ . Also, in the last column of Table 4.1 we have "Lie" for Lie algebras, "Malcev" for Malcev non-Lie algebras and "BL" for BL non-Malcev algebras.

The names of four-dimensional Lie algebras are from [5]. The classification of BL non-Lie algebras is taken from [22]. One of them is called  $g_3(\beta)$  here, since for  $\beta = 2$  we obtain the algebra  $g_3$  in the notation of [5]. The remaining BL non-Lie algebra is called  $g_6$ .

The names of five- and six-dimensional nilpotent Lie algebras are taken from [30]. The classification of Malcev non-Lie algebras of corresponding dimensions is deduced from [20]. We give names containing the letter "M" to these algebras. So in our notation a five- or six-dimensional algebra is Malcev and non-Lie iff it contains a letter "M" in its name, except the algebras  $M_6^1$  and  $M_2^1$  that correspond to the algebras  $g_{6,4}$  and  $g_{6,12}$  respectively in the notation of [30].

### 3. METHODS

In the present work we use the methods that were applied for Lie algebras in [7, 13, 14, 30]. First of all, it is well known that if  $A \rightarrow B$  and  $A \not\cong B$ , then  $\dim Der(A) < \dim Der(B)$ , where  $Der(A)$  is the algebra of derivations of  $A$ . We have computed the dimensions of algebras of derivations and have checked the assertion  $A \rightarrow B$  only for such  $A$  and  $B$  that  $\dim Der(A) < \dim Der(B)$ . Secondly, it is well known that if  $A \rightarrow C$  and  $C \rightarrow B$ , then  $A \rightarrow B$ . If there is no  $C$  such that  $A \rightarrow C$  and  $C \rightarrow B$ , then the assertion  $A \rightarrow B$  is called a *primary degeneration*. If  $\dim Der(A) < \dim Der(B)$  and there are no  $C$  and  $D$  such that  $C \rightarrow A$ ,  $B \rightarrow D$  and  $C \not\rightarrow D$ , then the assertion  $A \not\rightarrow B$  is called a *primary non-degeneration*. It is enough to prove only primary degenerations and non-degenerations to describe all degenerations in the variety under consideration. It is easy to see that any algebra degenerates to the algebra with zero multiplication.

Degenerations of four-dimensional and nilpotent five- and six-dimensional Lie algebras were described in [7, 13, 30]. Since the set  $\mathbb{L}(T)$  is closed for any  $T$ , a Lie algebra can't degenerate to a non-Lie algebra. So when we want to add Malcev or BL algebras to Lie algebras we don't have to check the degenerations from Lie algebras to any of the added algebras.

To prove the primary degenerations we construct the families of matrices parametrized by  $t$ . Namely, let  $A$  and  $B$  be two algebras represented by the structures  $\mu$  and  $\lambda$  from  $\mathbb{L}(T)$  respectively. Let  $e_1, \dots, e_n$  be a basis of  $V$ , for which  $\lambda$  is defined by structure constants  $c_{i,j}^k$  ( $1 \leq i, j, k \leq n$ ). If there exist  $a_i^j(t) \in \mathbb{C}$  ( $1 \leq i, j \leq n, t \in \mathbb{C}^*$ ) such that  $E_i^t = \sum_{j=1}^n a_i^j(t) e_j$  is a basis of  $V$  for  $t \in \mathbb{C}^*$  and the structure constants of  $\mu$  in the basis  $E_1^t, \dots, E_n^t$  are such polynomials  $c_{i,j}^k(t) \in \mathbb{C}[t]$  that  $c_{i,j}^k(0) = c_{i,j}^k$ , then  $A \rightarrow B$ . In this case  $E_1^t, \dots, E_n^t$  is called a *parametrized basis* for  $A \rightarrow B$ .

Tables 4.2 and 6.2 give parametrized bases for primary degenerations between four-dimensional BL algebras and six-dimensional nilpotent Malcev algebras respectively. These tables include all primary degenerations of the form  $A \rightarrow B$ , where  $A$  is a non-Lie algebra.

We now describe the methods for proving primary non-degenerations. The main tool for this is the following lemma.

**Lemma 1** ([4, 13]). *Let  $\mathcal{B}$  be a Borel subgroup of  $GL_n(\mathbb{C})$  and  $\mathcal{R} \subset \mathbb{L}(T)$  be a  $\mathcal{B}$ -stable closed subset. If  $A \rightarrow B$  and  $A$  can be represented by a structure  $\mu \in \mathcal{R}$ , then there is a structure  $\lambda \in \mathcal{R}$  representing  $B$ .*

Since any Borel subgroup of  $GL_n(\mathbb{C})$  is conjugate to the subgroup of upper triangular matrices, Lemma 1 can be applied in the following way. Let  $A$  and  $B$  be two algebras. Let  $\mu, \lambda$  be some structures in  $\mathbb{L}(T)$  representing  $A$  and  $B$  respectively. Suppose that there is a set of equations  $Q$  in variables  $x_{i,j}^k$  ( $1 \leq i, j, k \leq n$ ) such that if  $x_{i,j}^k = c_{i,j}^k$  ( $1 \leq i, j, k \leq n$ ) is a solution of all equations from  $Q$ , then  $\tilde{c}_{i,j}^k = c_{i,j}^k$  ( $1 \leq i, j, k \leq n$ ) is a solution for all equations from  $Q$  too in the following cases:

- (1) if  $\tilde{c}_{i,j}^k = \frac{\alpha_i \alpha_j}{\alpha_k} c_{i,j}^k$  for some  $\alpha_i \in \mathbb{C}^*$  ( $1 \leq i \leq n$ );
- (2) if there are some numbers  $1 \leq u < v \leq n$  and some  $\alpha \in \mathbb{C}$  such that

$$\tilde{c}_{i,j}^k = \begin{cases} c_{i,j}^k, & \text{if } i, j \neq u \text{ and } k \neq v, \\ c_{u,j}^k + \alpha c_{v,j}^k, & \text{if } i = u, j \neq u \text{ and } k \neq v, \\ c_{i,u}^k + \alpha c_{i,v}^k, & \text{if } i \neq u, j = u \text{ and } k \neq v, \\ c_{i,j}^v - \alpha c_{i,j}^u, & \text{if } i, j \neq u \text{ and } k = v, \\ c_{u,u}^k + \alpha(c_{v,u}^k + c_{u,v}^k) + \alpha^2 c_{v,v}^k, & \text{if } i = j = u \text{ and } k \neq v, \\ c_{u,j}^v + \alpha(c_{v,j}^v - c_{u,j}^u) - \alpha^2 c_{v,j}^u, & \text{if } i = u, j \neq u \text{ and } k = v, \\ c_{i,u}^v + \alpha(c_{i,v}^v - c_{i,u}^u) - \alpha^2 c_{i,u}^u, & \text{if } i \neq u, j = u \text{ and } k = v, \\ c_{u,u}^v + \alpha(c_{v,u}^v + c_{u,v}^v - c_{u,u}^u) + \alpha^2(c_{v,v}^v - c_{v,u}^u - c_{u,v}^u) - \alpha^3 c_{v,v}^u, & \text{if } i = j = u \text{ and } k = v. \end{cases}$$

Assume that there is a basis  $f_1, \dots, f_n$  of  $V$  such that the structure constants of  $\mu$  in this basis form a solution for all equations from  $Q$ , but there is no basis  $\tilde{f}_1, \dots, \tilde{f}_n$  of  $V$  such that the structure constants of  $\lambda$  in it form a solution for all equations from  $Q$ . Then  $A \not\rightarrow B$ .

We will often use two particular cases of Lemma 1. Firstly, if  $\dim A^l < \dim B^l$  for some  $l > 0$ , then  $A \not\rightarrow B$ . Secondly, if  $\dim Z_l(A) > \dim Z_l(B)$  for some  $l > 0$ , then  $A \not\rightarrow B$ . In the cases where these two criterions can't be applied, we define  $\mathcal{R}$  by some conditions, which can be expressed in terms of a set of equations  $Q$  satisfying the property described above, and give a basis for  $V$ , in which the structure constants of  $\mu$  satisfy all equations from  $Q$ . We omit everywhere the verification of the fact that  $Q$  satisfies the required conditions and the verification of the fact that structure constants of  $\lambda$  in any basis do not satisfy some equation from  $Q$ . These verifications can be done by direct calculations.

Another argument for the non-degeneration that we use is the so-called  $(i, j)$ -invariant. Given  $i, j > 0$ , we call  $c_{i,j} \in \mathbb{C}$  an  $(i, j)$ -invariant for the algebra  $A$  if

$$\text{tr}(\text{ad } x)^i \cdot \text{tr}(\text{ad } y)^j = c_{i,j} \text{tr}((\text{ad } x)^i \circ (\text{ad } y)^j)$$

for all  $x, y \in A$ . If  $c_{i,j}$  is an  $(i, j)$ -invariant for  $A$ , but at the same time it is not an  $(i, j)$ -invariant for  $B$ , then  $A \not\rightarrow B$ .

We give the proof of primary non-degenerations in Tables 4.3 and 6.3, where for each primary non-degeneration we give one of the arguments mentioned above.

If the number of orbits under the action of  $GL_n(\mathbb{C})$  on the variety  $\mathbb{L}(T)$  is finite, then the graph of primary degenerations gives the whole picture. In particular, the description of rigid algebras and irreducible components can be easily obtained. But in this work in some cases the situation is not so good. Then we have to be able to verify a little more complicated assertions. Let  $A_* = \{A_\alpha\}_{\alpha \in I}$  be a set of algebras and  $B$  be some other algebra. Suppose that  $A_\alpha$  is represented by the structure  $\mu_\alpha$  ( $\alpha \in I$ ) and  $B$  is represented by the structure  $\lambda$ . Then  $A_* \rightarrow B$  means  $\lambda \in \bigcup_{\alpha \in I} \overline{O(\mu_\alpha)}$ , and  $A_* \not\rightarrow B$  means  $\lambda \notin \bigcup_{\alpha \in I} \overline{O(\mu_\alpha)}$ .

Let  $A_*$ ,  $B$ ,  $\mu_\alpha$  ( $\alpha \in I$ ) and  $\lambda$  be as above. To prove that  $A_* \rightarrow B$  we have to construct a family of pairs  $(f(t), g(t))$  parametrized by  $t$ , where  $f(t) \in I$  and  $g(t) \in GL_n(\mathbb{C})$ . Namely, let  $e_1, \dots, e_n$  be a basis of  $V$ , for which  $\lambda$  is defined by structure constants  $c_{i,j}^k$  ( $1 \leq i, j, k \leq n$ ). If we construct  $a_i^j(t) \in \mathbb{C}$  ( $1 \leq i, j \leq n, t \in \mathbb{C}^*$ ) and  $f : \mathbb{C}^* \rightarrow I$  such that  $E_i^t = \sum_{j=1}^n a_i^j(t) e_j$  is a basis of  $V$  for  $t \in \mathbb{C}^*$  and the structure constants of  $\mu_{f(t)}$  in the basis  $E_1^t, \dots, E_n^t$  are such polynomials  $c_{i,j}^k(t) \in \mathbb{C}[t]$  that  $c_{i,j}^k(0) = c_{i,j}^k$ , then  $A_* \rightarrow B$ . In this case  $E_1^t, \dots, E_n^t$  and  $f(t)$  are called a parametrized basis and a parametrized index for  $A_* \rightarrow B$  respectively.

We now explain how to prove that  $A_* \not\rightarrow B$ . First of all, if  $\dim \text{Der}(A_\alpha) > \dim \text{Der}(B)$  for all  $\alpha \in I$ , then  $A_* \not\rightarrow B$ . One can use also the following generalization of Lemma 1, whose proof is the same as the proof of Lemma 1.

**Lemma 2.** *Let  $\mathcal{B}$  be a Borel subgroup of  $GL_n(\mathbb{C})$  and  $\mathcal{R} \subset \mathbb{L}(T)$  be a  $\mathcal{B}$ -stable closed subset. If  $A_* \rightarrow B$ , and for any  $\alpha \in I$  the algebra  $A_\alpha$  can be represented by a structure  $\mu_\alpha \in \mathcal{R}$ , then there is a structure  $\lambda \in \mathcal{R}$  representing  $B$ .*

#### 4. BINARY LIE ALGEBRAS OF DIMENSION 4

The following table contains the classification and some invariants of four-dimensional BL algebras. It collects results from [5, 22].

$A$	multiplication table	$Der(A)$	$Z(A)$	$A^2$	type
$n_3 \oplus \mathbb{C}$	$e_1 e_2 = e_3$	10	2	1	<i>Lie</i>
$n_4$	$e_1 e_2 = e_3, e_1 e_3 = e_4$	7	1	2	<i>Lie</i>
$r_2 \oplus \mathbb{C}^2$	$e_1 e_2 = e_2$	8	2	1	<i>Lie</i>
$r_2 \oplus r_2$	$e_1 e_2 = e_2, e_3 e_4 = e_4$	4	0	2	<i>Lie</i>
$sl_2 \oplus \mathbb{C}$	$e_1 e_2 = e_2, e_1 e_3 = -e_3, e_2 e_3 = e_1$	4	1	3	<i>Lie</i>
$g_1$	$e_1 e_2 = e_2, e_1 e_3 = e_3, e_1 e_4 = e_4$	12	0	3	<i>Lie</i>
$g_2(\beta)$	$e_1 e_2 = e_2, e_1 e_3 = e_3, e_1 e_4 = e_3 + \beta e_4$	8	0	$3, \beta \neq 0;$ $2, \beta = 0$	<i>Lie</i>
$g_3(\beta)$	$e_1 e_2 = e_2, e_1 e_3 = e_3,$ $e_1 e_4 = \beta e_4, e_2 e_3 = e_4$	7	0	3	<i>Lie</i> , for $\beta = 2$ , <i>Malcev</i> , for $\beta = -1$ <i>BL</i> , for $\beta \neq -1, 2$
$g_4(\alpha, \beta)$	$e_1 e_2 = e_2, e_1 e_3 = e_2 + \alpha e_3,$ $e_1 e_4 = e_3 + \beta e_4$	6	0	$3, \alpha \neq 0 \neq \beta;$ $2, \alpha \beta = 0$	<i>Lie</i>
$g_5(\alpha)$	$e_1 e_2 = e_2, e_1 e_3 = e_2 + \alpha e_3,$ $e_1 e_4 = (\alpha + 1)e_4, e_2 e_3 = e_4$	5	0	$3, \alpha \neq 0$ $2, \alpha = 0$	<i>Lie</i>
$g_6$	$e_1 e_2 = e_3, e_3 e_4 = e_3$	7	0	1	<i>BL</i>

Table 4.1. Binary Lie algebras of dimension 4.

The algebra  $g_4(\alpha_1, \beta_1)$  is isomorphic to  $g_4(\alpha_2, \beta_2)$  iff the proportions  $1 : \alpha_1 : \beta_1$  and  $1 : \alpha_2 : \beta_2$  coincide after some permutation. The algebra  $g_5(\alpha)$  is isomorphic to  $g_5(\beta)$  iff  $\alpha\beta = 1$  or  $\alpha = \beta$ . Apart from these two exceptions, any two algebras with different names from Table 4.1 are not isomorphic.

**Theorem 3.** The graph of primary degenerations for binary Lie algebras of dimension 4 has the following form:

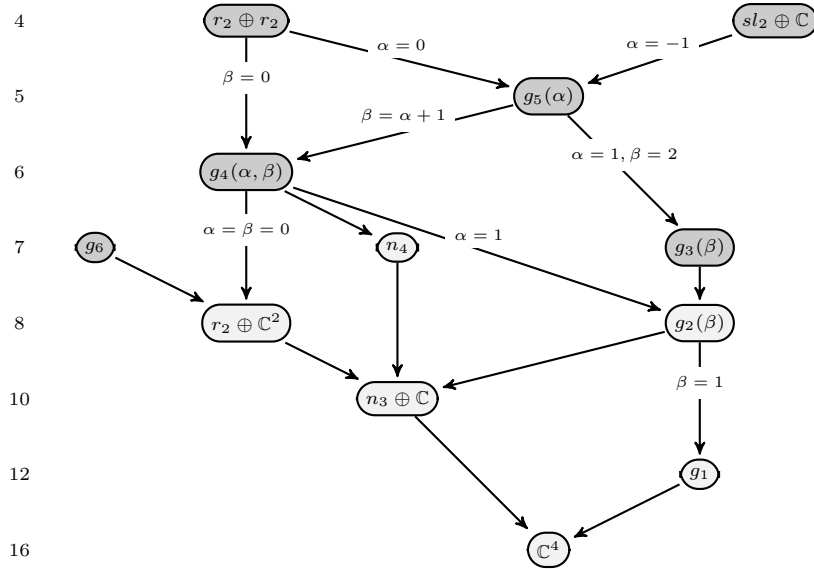


Figure I. The graph of primary degenerations for four-dimensional binary Lie algebras.

**Proof.** Tables 4.2 and 4.3 placed below give the proofs for all primary degenerations and non-degenerations including non-Lie algebras.

degenerations	parametrized bases
$g_3(\beta) \rightarrow g_2(\beta)$	$E_1^t = e_1 + e_2, E_2^t = te_2, E_3^t = (1 - \beta)e_3 + e_4, E_4^t = e_3 + e_4$
$g_6 \rightarrow r_2 \oplus \mathbb{C}^2$	$E_1^t = e_3, E_2^t = e_4, E_3^t = e_1, E_4^t = te_2$

Table 4.2. Degenerations of binary Lie algebras of dimension 4.

non-degenerations	arguments
$g_3(\beta) \not\rightarrow r_2 \oplus \mathbb{C}^2, g_1(\beta \neq 1), g_2(\gamma \neq \beta)$	$c_{ij}(g_3(\beta)) = \frac{(\beta^i+2)(\beta^j+2)}{\beta^{i+j+2}}$ , but $c_{ij}(r_2 \oplus \mathbb{C}^2) = 1, c_{ij}(g_1) = 3$ and $c_{ij}(g_2(\gamma)) = \frac{(\gamma^i+2)(\gamma^j+2)}{\gamma^{i+j+2}}$
$g_6 \not\rightarrow g_1, g_2(\beta)$	$\dim(g_6)^2 < \dim(g_2(\beta))^2 \leq \dim(g_1)^2$

Table 4.3. Non-degenerations of binary Lie algebras of dimension 4.

□

**Remark.** Gorbatsevich classified all finite-dimensional anticommutative algebras of level 3 in [12]. It follows from Theorem 3 that his classification is not correct. Namely, there are binary Lie algebras of level 3 in dimension 4, which are not included in the classification of Gorbatsevich.

**Corollary 4.**  $\text{Irr}(BL_4) = \{\mathcal{C}_i\}_{1 \leq i \leq 5}$ , where

$$\begin{aligned}\mathcal{C}_1 &= \overline{O(sl_2 \oplus \mathbb{C})} = O(\{sl_2 \oplus \mathbb{C}, g_5(-1), g_4(-1, 0), n_4, n_3 \oplus \mathbb{C}, \mathbb{C}^4\}), \\ \mathcal{C}_2 &= \overline{O(r_2 \oplus r_2)} = O\left(\{r_2 \oplus r_2, g_5(0), g_2(0), r_2 \oplus \mathbb{C}^2, n_4, n_3 \oplus \mathbb{C}, \mathbb{C}^4\} \cup \bigcup_{\alpha \in \mathbb{C}} \{g_4(\alpha, 0)\}\right), \\ \mathcal{C}_3 &= \bigcup_{\alpha \in \mathbb{C}} \overline{O(g_5(\alpha))} = O\left(\bigcup_{\alpha \in \mathbb{C}} \{g_5(\alpha), g_4(\alpha, \alpha + 1)\} \cup \{g_2(0), g_2(2), g_3(2), n_4, n_3 \oplus \mathbb{C}, \mathbb{C}^4\}\right), \\ \mathcal{C}_4 &= \bigcup_{\alpha, \beta \in \mathbb{C}} \overline{O(g_4(\alpha, \beta))} = O\left(\bigcup_{\alpha, \beta \in \mathbb{C}} \{g_4(\alpha, \beta), g_2(\beta)\} \cup \{r_2 \oplus \mathbb{C}^2, n_4, n_3 \oplus \mathbb{C}, g_1, \mathbb{C}^4\}\right), \\ \mathcal{C}_5 &= \bigcup_{\beta \in \mathbb{C}} \overline{O(g_3(\beta))} = O\left(\bigcup_{\beta \in \mathbb{C}} \{g_3(\beta), g_2(\beta)\} \cup \{g_6, r_2 \oplus \mathbb{C}^2, n_3 \oplus \mathbb{C}, g_1, \mathbb{C}^4\}\right).\end{aligned}$$

In particular,  $\text{Rig}(BL_4) = \text{Rig}(Lie_4) = \{sl_2 \oplus \mathbb{C}, r_2 \oplus r_2\}$ .

**Proof.** In view of Theorem 3 and the fact that  $Lie_4$  is a closed subset of  $BL_4$  it is enough to prove that

$$\begin{aligned}g_5(*) \not\rightarrow g_4(\alpha, \beta) (g_4(\alpha, \beta) \not\cong g_4(\gamma, \gamma + 1) \text{ for any } \gamma \in \mathbb{C}), g_5(*) \not\rightarrow r_2 \oplus \mathbb{C}^2, g_5(*) \not\rightarrow g_2(\beta) (\beta \neq 0, 2), g_5(*) \not\rightarrow g_1, \\ g_4(*, *) \not\rightarrow g_3(2), g_3(*) \rightarrow g_6, g_3(*) \not\rightarrow n_4,\end{aligned}$$

where  $g_5(*) = \{g_5(\alpha)\}_{\alpha \in \mathbb{C}}$ ,  $g_4(*, *) = \{g_4(\alpha, \beta)\}_{\alpha, \beta \in \mathbb{C}}$  and  $g_3(*) = \{g_3(\beta)\}_{\beta \in \mathbb{C}}$ . Let us define

$$\mathcal{R} = \left\{ A \left| \begin{array}{l} A = \langle f_1, f_2, f_3, f_4 \rangle, \langle f_3, f_4 \rangle^2 = 0, \langle f_2, f_3, f_4 \rangle^2 \subset \langle f_4 \rangle, A\langle f_2, f_3, f_4 \rangle \subset \langle f_2, f_3, f_4 \rangle, A\langle f_3, f_4 \rangle \subset \langle f_3, f_4 \rangle, \\ A\langle f_4 \rangle \subset \langle f_4 \rangle, c_{1,2}^2 + c_{1,3}^3 = c_{1,4}^4, \text{ where } f_i f_j = \sum_{k=1}^4 c_{i,j}^k f_k \text{ for all } 1 \leq i, j \leq 4 \end{array} \right. \right\}.$$

One can take  $f_1 = e_1, f_2 = e_3, f_3 = e_2$  and  $f_4 = e_4$  and check that  $g_5(\alpha) \in \mathcal{R}$  for all  $\alpha \in \mathbb{C}$ .

Let us prove that  $g_2(\beta) \notin \mathcal{R}$  if  $\beta \neq 0, 2$ . Assume that there is some basis  $\tilde{f}_i$  ( $1 \leq i \leq 4$ ) of  $V$  such that the structure constants  $\tilde{c}_{i,j}^k$  of  $g_2(\beta)$  in it satisfy all required conditions. Let  $U = \langle \tilde{f}_2, \tilde{f}_3, \tilde{f}_4 \rangle$  and  $L : U \rightarrow U$  be the operator of left multiplication by  $\tilde{f}_1$ . It follows from the definition of  $\mathcal{R}$  that the matrix of  $L$  in the basis  $\tilde{f}_2, \tilde{f}_3, \tilde{f}_4$  is lower triangular. Hence,  $\tilde{c}_{1,2}^2, \tilde{c}_{1,3}^3$  and  $\tilde{c}_{1,4}^4$  are eigen values of  $L$ . On the other hand, it is easy to see that  $U = \langle e_2, e_3, e_4 \rangle$  and  $\tilde{f}_1 = ce_1 + v$  for some  $c \in \mathbb{C}^*$  and  $v \in U$ . Then the eigen values of  $L$  are  $c, c$  and  $\beta c$ . Then we have  $c = (\beta + 1)c$  or  $\beta c = 2c$ , i.e.  $\beta = 0$  or  $\beta = 2$ .

Analogously one can prove that  $g_4(\alpha, \beta) \notin \mathcal{R}$  if  $\alpha - \beta \neq 1, \alpha - \beta \neq -1$  and  $\alpha + \beta \neq 1$ , and  $r_2 \oplus \mathbb{C}^2, g_1 \notin \mathcal{R}$ .

Since  $\langle e_2, e_3, e_4 \rangle$  is an abelian subalgebra of  $g_4(\alpha, \beta)$  and there is no three-dimensional abelian subalgebra in  $g_3(2)$ , we have  $g_4(*, *) \not\rightarrow g_3(2)$  by Lemma 2. Let now define

$$\mathcal{R} = \left\{ A \left| \begin{array}{l} A = \langle f_1, f_2, f_3, f_4 \rangle, \langle f_2, f_3, f_4 \rangle^2 \subset \langle f_4 \rangle, A\langle f_2, f_3, f_4 \rangle \subset \langle f_2, f_3, f_4 \rangle, A\langle f_3, f_4 \rangle \subset \langle f_3, f_4 \rangle, A\langle f_4 \rangle \subset \langle f_4 \rangle, \\ c_{1,2}^2 = c_{1,3}^3, c_{1,2}^3 = 0, \text{ where } f_i f_j = \sum_{k=1}^4 c_{i,j}^k f_k \text{ for all } 1 \leq i, j \leq 4 \end{array} \right. \right\}.$$

One can take  $f_i = e_i$  ( $1 \leq i \leq 4$ ) and check that  $g_3(\beta) \in \mathcal{R}$  for all  $\beta \in \mathbb{C}$ . On the other hand, it is not hard to check that  $n_4 \notin \mathcal{R}$ . Finally, to prove that  $g_3(*) \rightarrow g_6$  it is enough to take the parametrized basis

$$E_1^t = e_2, E_2^t = e_3, E_3^t = e_4, E_4^t = -te_1$$

and the parametrized index  $\beta(t) = \frac{1}{t}$ .

□

**Corollary 5.**  $\text{Irr}(Mal_4) = \{\mathcal{C}_i\}_{1 \leq i \leq 4} \cup \{\mathcal{C}'_5\}$ , where  $\mathcal{C}_i$  ( $1 \leq i \leq 4$ ) are the same as in Corollary 4, and

$$\mathcal{C}'_5 = \overline{O(g_3(-1))} = O(\{g_3(-1), g_2(-1), n_3 \oplus \mathbb{C}, \mathbb{C}^4\}).$$

In particular,  $\text{Rig}(Mal_4) = \{sl_2 \oplus \mathbb{C}, r_2 \oplus r_2, g_3(-1)\}$ .

**Proof.** Everything follows from Theorem 3, Corollary 4 and Table 4.1.

□

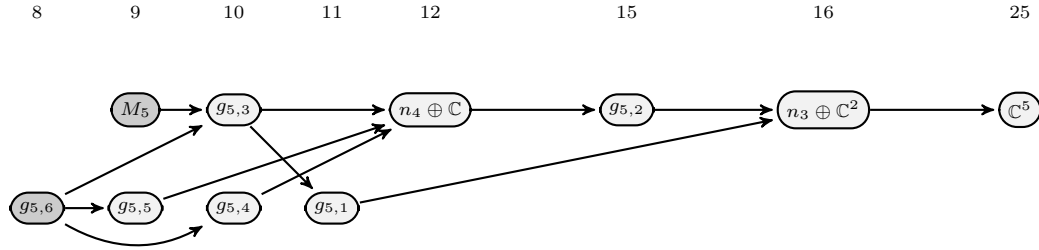
## 5. DEGENERATIONS OF NILPOTENT MALCEV ALGEBRAS OF DIMENSION 5

For five-dimensional nilpotent Malcev algebras we have the following table, which is constructed using results of [13] and [20].

$A$	multiplication table	$\text{Der}(A)$	$Z_l(A)$	$A^2$	$A^3$
$n_3 \oplus \mathbb{C}^2$	$e_1 e_2 = e_3$	16	$2 + 13$	1	0
$n_4 \oplus \mathbb{C}$	$e_1 e_2 = e_3, e_1 e_3 = e_4$	12	$1 + 124$	2	1
$g_{5,1}$	$e_1 e_2 = e_5, e_3 e_4 = e_5$	11	15	1	0
$g_{5,2}$	$e_1 e_2 = e_4, e_1 e_3 = e_5$	15	25	2	0
$g_{5,3}$	$e_1 e_2 = e_3, e_1 e_4 = e_5, e_2 e_3 = e_5$	10	135	2	1
$g_{5,4}$	$e_1 e_2 = e_3, e_1 e_3 = e_4, e_2 e_3 = e_5$	10	235	3	2
$g_{5,5}$	$e_1 e_2 = e_3, e_1 e_3 = e_4, e_1 e_4 = e_5$	9	1235	3	2
$g_{5,6}$	$e_1 e_2 = e_3, e_1 e_3 = e_4, e_1 e_4 = e_5, e_2 e_3 = e_5$	8	1235	3	2
$M_5$	$e_1 e_2 = e_4, e_3 e_4 = e_5$	9	135	2	0

Table 5.1. Nilpotent Malcev algebras of dimension 5.

**Theorem 6.** *The graph of primary degenerations for nilpotent Malcev algebras of dimension 5 has the following form:*



**Figure II.** The graph of primary degenerations for five-dimensional nilpotent Malcev algebras.

**Proof.** It is enough to verify the assertions of the form  $M_5 \rightarrow A$  for such  $A$  that  $\dim \text{Der}(A) < 9$ . So we have to check that  $M_5 \rightarrow g_{5,3}$  and  $M_5 \not\rightarrow g_{5,4}$ . The parametrized basis formed by  $E_1^t = e_1 - e_4$ ,  $E_2^t = te_2 + te_3$ ,  $E_3^t = te_4 + te_5$ ,  $E_4^t = t^2 e_3$  and  $E_5^t = t^2 e_5$  gives the required degeneration. The assertion  $M_5 \not\rightarrow g_{5,4}$  follows from the fact that  $\dim (g_{5,4})^2 > \dim (M_5)^2$ . □

**Corollary 7.**  $\text{Irr}(N\text{Mal}_5) = \{\mathcal{C}_1, \mathcal{C}_2\}$ , where  $\mathcal{C}_1 = \overline{O(g_{5,6})} = N\text{Lie}_5$  and  $\mathcal{C}_2 = \overline{O(M_5)} = N\text{Mal}_5 \setminus \{g_{5,6}, g_{5,5}, g_{5,4}\}$ . In particular,  $\text{Rig}(N\text{Mal}_5) = \{g_{5,6}, M_5\}$ .

**Proof.** Since there is only finite number of isomorphism classes of five-dimensional nilpotent Malcev algebras, everything follows from Theorem 6. □

## 6. DEGENERATIONS OF NILPOTENT MALCEV ALGEBRAS OF DIMENSION 6

We use the table of invariants for nilpotent six-dimensional Lie algebras from [30] and classification of nilpotent six-dimensional Malcev non-Lie algebras from [20] to construct the table containing important invariants for nilpotent six-dimensional Malcev algebras. To simplify the notation we write  $g_i$  instead of  $g_{6,i}$ , and  $g_i^{\mathbb{C}}$  and  $M_5^{\mathbb{C}}$  instead of  $g_{5,i} \oplus \mathbb{C}$  and  $M_5 \oplus \mathbb{C}$  respectively.

$A$	multiplication table	$Der(A)$	$Z_l(A)$	$A^2$	$A^3$
$g_1$	$e_1e_2 = e_3, e_1e_3 = e_4, e_1e_4 = e_6, e_2e_3 = e_6, e_2e_5 = e_6$	11	1346	3	2
$g_2$	$e_1e_2 = e_3, e_1e_3 = e_4, e_1e_4 = e_6, e_2e_5 = e_6$	12	1346	3	2
$g_3$	$e_1e_2 = e_3, e_1e_3 = e_6, e_4e_5 = e_6$	14	146	2	1
$g_5$	$e_1e_2 = e_3, e_1e_3 = e_4, e_1e_4 = e_5, e_1e_5 = e_6, e_2e_3 = e_5, e_2e_4 = e_6$	9	12346	4	3
$g_6$	$e_1e_2 = e_3, e_1e_3 = e_4, e_1e_4 = e_5, e_2e_3 = e_5, e_2e_5 = e_6, e_3e_4 = -e_6$	8	12346	4	3
$g_7$	$e_1e_2 = e_3, e_1e_3 = e_4, e_1e_4 = e_5, e_1e_5 = e_6, e_2e_3 = e_6$	10	12346	4	3
$g_8$	$e_1e_2 = e_3, e_1e_3 = e_4, e_2e_5 = e_6, e_3e_4 = -e_6$	9	12346	3	2
$g_9$	$e_1e_2 = e_3, e_1e_3 = e_4, e_1e_4 = e_5, e_1e_5 = e_6$	11	12346	4	3
$g_{10}$	$e_1e_2 = e_4, e_1e_3 = e_5, e_1e_4 = e_6, e_3e_5 = e_6$	12	136	3	1
$g_{14}$	$e_1e_2 = e_3, e_1e_3 = e_4, e_1e_5 = e_6, e_2e_3 = e_5, e_2e_4 = e_6$	10	1346	4	3
$g_{15}$	$e_1e_2 = e_3, e_1e_3 = e_5, e_1e_4 = e_6, e_2e_3 = e_6$	13	246	3	2
$g_{16}$	$e_1e_2 = e_3, e_1e_3 = e_5, e_1e_4 = e_5, e_2e_3 = e_6$	12	246	3	2
$g_{17}$	$e_1e_2 = e_3, e_1e_3 = e_5, e_1e_4 = e_6$	15	246	3	1
$g_{18}$	$e_1e_2 = e_3, e_1e_3 = e_5, e_2e_4 = e_6$	13	246	3	1
$g_{20}$	$e_1e_2 = e_3, e_1e_3 = e_5, e_1e_4 = e_6, e_2e_4 = e_5$	14	246	3	1
$g_{21}$	$e_1e_2 = e_5, e_1e_3 = e_6, e_3e_4 = e_5$	17	26	2	0
$g_{23}$	$e_1e_2 = e_3, e_1e_3 = e_4, e_1e_4 = e_5, e_2e_3 = e_6$	11	2346	4	3
$g_{24}$	$e_1e_2 = e_4, e_1e_3 = e_5, e_2e_3 = e_6$	16	36	3	0
$g_1^{\mathbb{C}}$	$e_1e_2 = e_5, e_3e_4 = e_5$	21	$1 + 15$	1	0
$g_2^{\mathbb{C}}$	$e_1e_2 = e_4, e_1e_3 = e_5$	19	$1 + 25$	2	0
$g_3^{\mathbb{C}}$	$e_1e_2 = e_3, e_1e_4 = e_5, e_2e_3 = e_5$	15	$1 + 135$	2	1
$g_4^{\mathbb{C}}$	$e_1e_2 = e_3, e_1e_3 = e_4, e_2e_3 = e_5$	15	$1 + 235$	3	2
$g_5^{\mathbb{C}}$	$e_1e_2 = e_3, e_1e_3 = e_4, e_1e_4 = e_5$	13	$1 + 1235$	3	2
$g_6^{\mathbb{C}}$	$e_1e_2 = e_3, e_1e_3 = e_4, e_1e_4 = e_5, e_2e_3 = e_5$	12	$1 + 1235$	3	2
$n_3 \oplus n_3$	$e_1e_3 = e_5, e_2e_4 = e_6$	16	$13 + 13$	2	0
$n_4 \oplus \mathbb{C}^2$	$e_1e_2 = e_3, e_1e_3 = e_4$	17	$2 + 124$	2	1
$n_3 \oplus \mathbb{C}^3$	$e_1e_2 = e_3$	24	$3 + 13$	1	0
$M_5^{\mathbb{C}}$	$e_1e_2 = e_5, e_3e_5 = e_6$	14	$1 + 135$	2	1
$M_1^{0,1}$	$e_1e_2 = e_5, e_3e_4 = e_5, e_3e_5 = e_6$	13	126	2	1
$M_1^{1,0}$	$e_1e_2 = e_5, e_1e_4 = e_6, e_3e_5 = e_6$	12	136	2	1
$M_1^{1,1}$	$e_1e_2 = e_5, e_1e_4 = e_6, e_3e_4 = e_5, e_3e_5 = e_6$	11	126	2	1
$M_2^0$	$e_1e_2 = e_4, e_1e_3 = e_5, e_2e_5 = e_6$	12	246	3	1
$M_2^{-1}$	$e_1e_2 = e_4, e_1e_3 = e_5, e_2e_5 = e_6, e_3e_4 = -e_6$	13	136	3	1
$M_2^{\epsilon}, \epsilon \neq -1, 0$	$e_1e_2 = e_4, e_1e_3 = e_5, e_2e_5 = e_6, e_3e_4 = \epsilon e_6$	11	136	3	1
$M_3$	$e_1e_2 = e_4, e_1e_3 = e_5, e_2e_4 = e_6, e_2e_5 = e_6, e_3e_4 = -e_6$	11	136	3	1
$M_4$	$e_1e_2 = e_4, e_1e_3 = e_5, e_1e_5 = e_6, e_3e_4 = e_6,$	13	136	3	1
$M_5^0$	$e_1e_2 = e_4, e_2e_4 = e_5, e_3e_4 = e_6$	11	246	3	2
$M_5^1$	$e_1e_2 = e_4, e_1e_3 = e_5, e_2e_4 = e_5, e_3e_4 = e_6$	10	246	3	2
$M_6^0$	$e_1e_2 = e_3, e_1e_3 = e_5, e_1e_5 = e_6, e_3e_4 = e_6$	10	1346	3	2
$M_6^{\epsilon}, \epsilon \neq 0$	$e_1e_2 = e_3, e_1e_3 = e_5, e_1e_5 = e_6, e_2e_4 = \epsilon e_5, e_3e_4 = e_6$	10	1236	3	2
$M_7^0$	$e_1e_2 = e_4, e_1e_4 = e_5, e_1e_5 = e_6, e_2e_3 = e_5$	11	1246	3	2
$M_7^1$	$e_1e_2 = e_4, e_1e_4 = e_5, e_1e_5 = e_6, e_2e_3 = e_5, e_2e_4 = e_6$	10	1246	3	2

Table 6.1. Nilpotent Malcev algebras of dimension 6.

The algebra  $M_2^{\epsilon}$  is isomorphic to  $M_2^{\epsilon'}$  iff  $\epsilon\epsilon' = 1$  or  $\epsilon = \epsilon'$ . Apart from this exception any two algebras with different names from Table 6.1 are not isomorphic.

**Theorem 8.** *The graph of primary degenerations for nilpotent Malcev algebras of dimension 6 has the form presented in Figure III.*

**Proof.** Tables 6.2 and 6.3 presented below give the proofs for all primary degenerations and non-degenerations including non-Lie algebras.

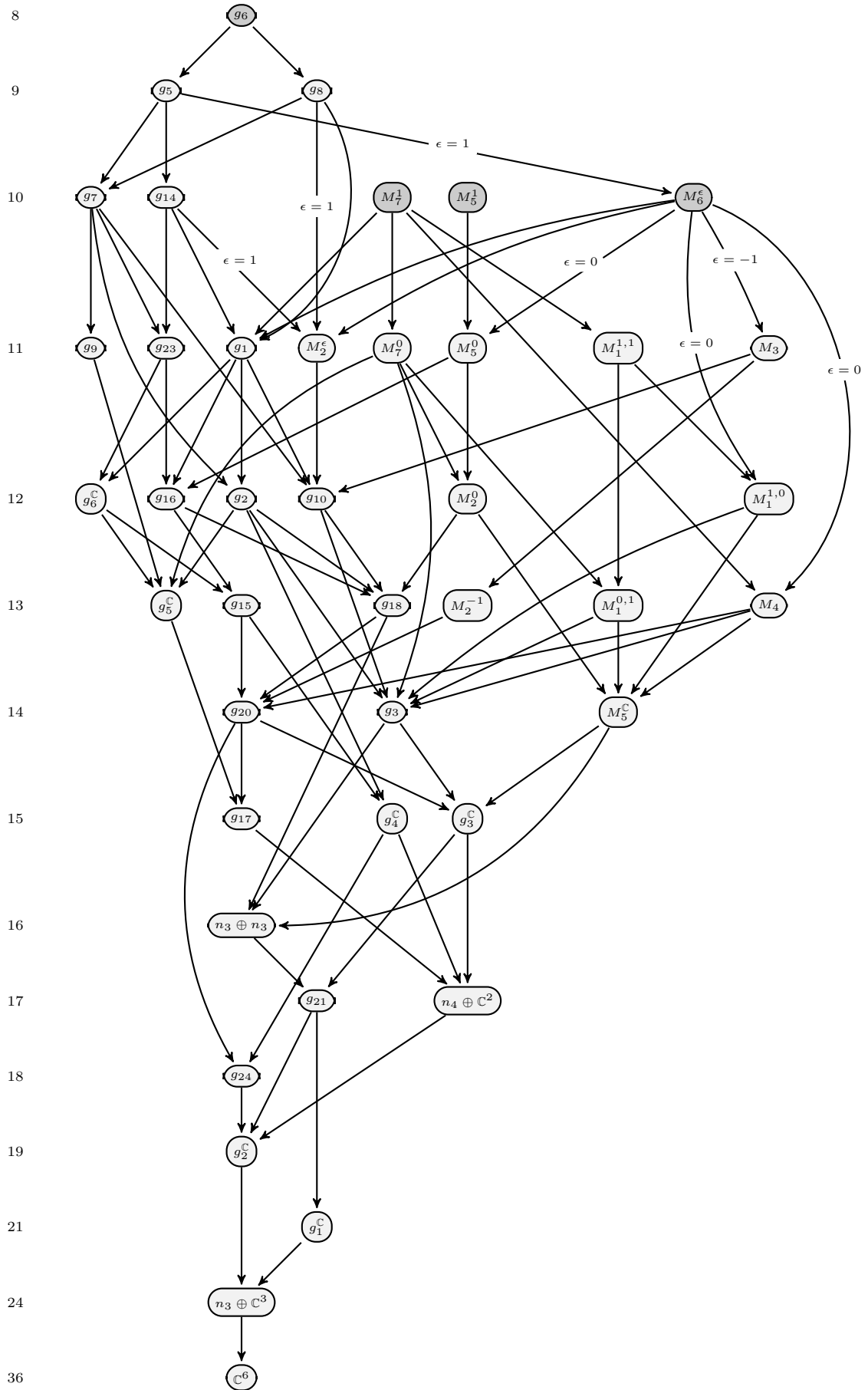
degenerations	parametrized bases
$M_7^1 \rightarrow g_1$	$E_1^t = e_1 - \frac{e_2}{t} - \frac{e_3}{t}, E_2^t = e_2, E_3^t = e_4 + \frac{e_5}{t}, E_4^t = e_5,$ $E_5^t = te_3 + e_4 + \frac{e_5}{t}, E_6^t = e_6$
$M_7^1 \rightarrow M_7^0$	$E_1^t = e_1, E_2^t = te_2, E_3^t = e_3, E_4^t = te_4, E_5^t = te_5, E_6^t = te_6$
$M_7^1 \rightarrow M_4$	$E_1^t = e_2, E_2^t = -e_3, E_3^t = -te_1, E_4^t = -e_5, E_5^t = te_4, E_6^t = te_6$
$M_7^1 \rightarrow M_1^{1,1}$	$E_1^t = e_2, E_2^t = \frac{e_3}{t}, E_3^t = e_1, E_4^t = \frac{e_4}{t}, E_5^t = \frac{e_5}{t}, E_6^t = \frac{e_6}{t}$
$M_5^1 \rightarrow M_5^0$	$E_1^t = e_1, E_2^t = e_2, E_3^t = te_3, E_4^t = e_4, E_5^t = e_5, E_6^t = te_6$
$M_6^\epsilon \rightarrow g_1$	$E_1^t = e_1 + \frac{e_4}{2t}, E_2^t = e_2 + \frac{(\epsilon+2)e_3}{2t} - e_4, E_3^t = e_3 + \frac{e_5}{t} - \frac{(\epsilon+2)e_6}{4t^2},$ $E_4^t = e_5 + \frac{e_6}{2t}, E_5^t = te_4, E_6^t = e_6$
$M_6^\epsilon \rightarrow M_2^\epsilon (\epsilon \neq 0)$	$E_1^t = e_2, E_2^t = \frac{e_4}{t}, E_3^t = e_1, E_4^t = \frac{e_5}{t}, E_5^t = -e_3, E_6^t = \frac{e_6}{t}$
$M_6^\epsilon \rightarrow M_5^0 (\epsilon = 0)$	$E_1^t = -e_2, E_2^t = e_1, E_3^t = -\frac{e_4}{t}, E_4^t = e_3, E_5^t = e_5, E_6^t = \frac{e_6}{t}$
$M_6^\epsilon \rightarrow M_1^{1,0} (\epsilon = 0)$	$E_1^t = te_1, E_2^t = e_2, E_3^t = -e_4, E_4^t = e_5, E_5^t = te_3, E_6^t = te_6$
$M_6^\epsilon \rightarrow M_4 (\epsilon = 0)$	$E_1^t = e_1, E_2^t = e_2, E_3^t = \frac{e_3}{t} - \frac{e_4}{t}, E_4^t = e_4, E_5^t = \frac{e_5}{t}, E_6^t = \frac{e_6}{t}$
$M_6^\epsilon \rightarrow M_3 (\epsilon = -1)$	$E_1^t = -e_2 - \frac{e_3}{2}, E_2^t = te_1, E_3^t = te_4, E_4^t = te_3 + \frac{te_5}{2},$ $E_5^t = te_5 - \frac{te_6}{2}, E_6^t = t^2e_6$
$M_2^\epsilon \rightarrow g_{10} (\epsilon \neq -1, 0)$	$E_1^t = \frac{\epsilon e_1}{t} + e_2 - e_3, E_2^t = te_1, E_3^t = \frac{(\epsilon+1)te_3}{\epsilon} - e_1, E_4^t = (\epsilon+1)te_5,$ $E_5^t = e_4 + \epsilon e_5, E_6^t = (\epsilon+1)te_6$
$M_2^\epsilon \rightarrow g_{18} (\epsilon = 0)$	$E_1^t = e_2 + e_3, E_2^t = e_1, E_3^t = -e_4 - e_5, E_4^t = te_3, E_5^t = -e_6, E_6^t = te_5$
$M_2^\epsilon \rightarrow M_5^C (\epsilon = 0)$	$E_1^t = e_1, E_2^t = e_3, E_3^t = e_2, E_4^t = \frac{e_4}{t}, E_5^t = e_5, E_6^t = e_6$
$M_2^\epsilon \rightarrow g_{20} (\epsilon = -1)$	$E_1^t = e_1 - e_2, E_2^t = e_3, E_3^t = e_5, E_4^t = te_2 + e_4, E_5^t = -e_6, E_6^t = te_4$
$M_7^0 \rightarrow g_5^C$	$E_1^t = e_1, E_2^t = e_2, E_3^t = e_4, E_4^t = e_5, E_5^t = e_6, E_6^t = te_3$
$M_7^0 \rightarrow g_3$	$E_1^t = te_1, E_2^t = \frac{e_4}{t^2} - \frac{e_5}{t^2}, E_3^t = \frac{e_5}{t} - \frac{e_6}{t}, E_4^t = e_2, E_5^t = e_3, E_6^t = e_6$
$M_7^0 \rightarrow M_2^0$	$E_1^t = e_2, E_2^t = e_1, E_3^t = \frac{e_3}{t}, E_4^t = -e_4, E_5^t = \frac{e_5}{t}, E_6^t = \frac{e_6}{t}$
$M_7^0 \rightarrow M_1^{0,1}$	$E_1^t = te_3, E_2^t = -\frac{e_2}{t}, E_3^t = te_1, E_4^t = \frac{e_4}{t}, E_5^t = e_5, E_6^t = te_6$
$M_5^0 \rightarrow g_{16}$	$E_1^t = te_2, E_2^t = -e_1 - e_3 - e_4, E_3^t = te_4 - te_5, E_4^t = te_3 + te_4, E_5^t = t^2e_5, E_6^t = -te_6$
$M_5^0 \rightarrow M_2^0$	$E_1^t = e_1 + e_2, E_2^t = e_3 + e_4, E_3^t = te_2, E_4^t = e_5, E_5^t = te_4, E_6^t = te_6$
$M_1^{1,1} \rightarrow M_1^{0,1}$	$E_1^t = te_1, E_2^t = e_2, E_3^t = e_3, E_4^t = te_4, E_5^t = te_5, E_6^t = te_6$
$M_1^{1,1} \rightarrow M_1^{1,0}$	$E_1^t = e_1, E_2^t = e_2, E_3^t = te_3, E_4^t = te_4, E_5^t = e_5, E_6^t = te_6$
$M_3 \rightarrow g_{10}$	$E_1^t = e_1 + e_2 + e_3, E_2^t = t^2e_2 + t^2e_3, E_3^t = te_2, E_4^t = t^2e_4 + t^2e_5, E_5^t = te_4, E_6^t = t^2e_6$
$M_3 \rightarrow M_2^{-1}$	$E_1^t = e_1, E_2^t = te_2, E_3^t = e_3, E_4^t = te_4, E_5^t = e_5, E_6^t = te_6$
$M_1^{1,0} \rightarrow g_3$	$E_1^t = e_1 + e_3, E_2^t = te_2, E_3^t = te_5, E_4^t = te_3, E_5^t = -e_4 + e_5, E_6^t = te_6$
$M_1^{1,0} \rightarrow M_5^C$	$E_1^t = e_1, E_2^t = e_2, E_3^t = e_3, E_4^t = te_4, E_5^t = e_5, E_6^t = e_6$
$M_1^{0,1} \rightarrow g_3$	$E_1^t = te_3, E_2^t = \frac{e_4}{t^2} - \frac{e_5}{t^2}, E_3^t = \frac{e_5}{t} - \frac{e_6}{t}, E_4^t = e_1, E_5^t = e_2, E_6^t = e_6$
$M_1^{0,1} \rightarrow M_5^C$	$E_1^t = e_1, E_2^t = e_2, E_3^t = e_3, E_4^t = te_4, E_5^t = e_5, E_6^t = e_6$
$M_4 \rightarrow g_{20}$	$E_1^t = \frac{e_1}{t}, E_2^t = e_3, E_3^t = \frac{e_5}{t}, E_4^t = e_2 + \frac{e_4}{t^2}, E_5^t = \frac{e_6}{t^2}, E_6^t = \frac{e_4}{t}$
$M_4 \rightarrow g_3$	$E_1^t = e_1 + e_3, E_2^t = e_2, E_3^t = e_4, E_4^t = te_3, E_5^t = \frac{e_4}{t} - \frac{e_5}{t}, E_6^t = e_6$
$M_4 \rightarrow M_5^C$	$E_1^t = te_1, E_2^t = \frac{e_2}{t}, E_3^t = e_3, E_4^t = e_5, E_5^t = e_4, E_6^t = e_6$
$M_5^C \rightarrow g_3^C$	$E_1^t = e_1 - e_4, E_2^t = te_2 + te_3, E_3^t = te_4 + te_5, E_4^t = t^2e_3, E_5^t = t^2e_5, E_6^t = e_6$
$M_5^C \rightarrow n_3 \oplus n_3$	$E_1^t = e_1, E_2^t = e_3, E_3^t = te_2, E_4^t = e_4 + e_5, E_5^t = te_5, E_6^t = e_6$

Table 6.2. Degenerations of nilpotent Malcev algebras of dimension 6.



non-degenerations	arguments
$M_7^1 \not\rightarrow g_9, g_{23}$	$\dim(g_9)^2 = \dim(g_{23})^2 > \dim(M_7^1)^2$
$M_7^1 \not\rightarrow M_2^\epsilon (\epsilon \neq 0), M_5^0$	$\mathcal{R} = \left\{ A \mid \begin{array}{l} A = \langle f_1, f_2, f_3, f_4, f_5, f_6 \rangle, A^2 \subset \langle f_4, f_5, f_6 \rangle, \\ \langle f_3, f_4, f_5, f_6 \rangle^2 = 0, A\langle f_3, f_4, f_5, f_6 \rangle \subset \langle f_5, f_6 \rangle \end{array} \right\}$ $M_7^1 \in \mathcal{R}$ (take $f_i = e_i$ for $1 \leq i \leq 6$ ), but $M_2^\epsilon, M_5^0 \notin \mathcal{R}$
$M_5^1 \not\rightarrow \left\{ \begin{array}{l} M_2^\epsilon (\epsilon \neq 0), M_7^0, M_1^{1,0}, \\ M_1^{0,1}, M_4, g_3 \end{array} \right\}$	$\dim Z(M_2^\epsilon) = \dim Z(M_7^0) = \dim Z(M_1^{1,0})$ $= \dim Z(M_1^{0,1}) = \dim Z(M_4) = \dim Z(g_3) < \dim Z(M_5^1)$
$M_5^1 \not\rightarrow g_5^C$	$\dim Z_2(g_5^C) < \dim Z_2(M_5^1)$
$M_6^\epsilon \not\rightarrow g_9, g_{23}$	$\dim(g_9)^2 = \dim(g_{23})^2 > \dim(M_6^\epsilon)^2$
$M_6^\epsilon \not\rightarrow M_1^{0,1} (\epsilon = 0)$	$\dim Z_2(M_1^{0,1}) < \dim Z_2(M_6^0)$
$M_6^\epsilon \not\rightarrow \left\{ \begin{array}{l} M_2^{\epsilon'} (M_2^{\epsilon'} \not\cong M_2^\epsilon), \\ M_5^C (\epsilon \neq 0) \end{array} \right\}$	$\mathcal{R} = \left\{ A \mid \begin{array}{l} A = \langle f_1, f_2, f_3, f_4, f_5, f_6 \rangle, A^2 \subset \langle f_4, f_5, f_6 \rangle, \\ \langle f_3, f_4, f_5, f_6 \rangle^2 = 0, \\ x(yz) = \epsilon y(xz) \forall x \in A, y, z \in \langle f_2, f_3, f_4, f_5, f_6 \rangle \end{array} \right\}$ $M_6^\epsilon \in \mathcal{R}$ (take $f_1 = e_1, f_2 = e_4, f_3 = e_2, f_4 = e_3, f_5 = e_5$ and $f_6 = e_6$ ), but $M_2^{\epsilon'} \notin \mathcal{R}$ and, if $\epsilon \neq 0$ , then $M_5^C \notin \mathcal{R}$
$M_2^\epsilon \not\rightarrow g_5^C, g_4^C$	$\dim(g_5^C)^3 = \dim(g_4^C)^3 > \dim(M_2^\epsilon)^3$
$M_2^\epsilon \not\rightarrow n_3 \oplus n_3 (\epsilon = -1)$	$\mathcal{R} = \left\{ A \mid \begin{array}{l} A = \langle f_1, f_2, f_3, f_4, f_5, f_6 \rangle, A^2 \subset \langle f_4, f_5, f_6 \rangle, \\ \langle f_3, f_4, f_5, f_6 \rangle^2 = 0, A\langle f_4, f_5, f_6 \rangle \subset \langle f_6 \rangle, \\ \langle f_2, f_3, f_4, f_5, f_6 \rangle \langle f_5, f_6 \rangle = 0, \\ \langle f_2, f_3, f_4, f_5, f_6 \rangle \langle f_3, f_4, f_5, f_6 \rangle \subset \langle f_5, f_6 \rangle, \\ c_{2,3}^5 c_{1,5}^6 + c_{2,4}^6 c_{1,3}^4 = 0, c_{2,3}^5 c_{1,4}^6 = c_{2,4}^6 c_{1,3}^5, \\ \text{where } f_i f_j = \sum_{k=1}^6 c_{i,j}^k f_k \text{ for all } 1 \leq i, j \leq 6 \end{array} \right\}$ $M_2^{-1} \in \mathcal{R}$ (take $f_1 = e_2, f_2 = e_3, f_3 = e_1, f_4 = e_4, f_5 = e_5$ and $f_6 = e_6$ ), but $n_3 \oplus n_3 \notin \mathcal{R}$
$M_7^0 \not\rightarrow g_{10}, M_1^{1,0}, M_4, g_4^C$	$\mathcal{R} = \left\{ A \mid \begin{array}{l} A = \langle f_1, f_2, f_3, f_4, f_5, f_6 \rangle, A^2 \subset \langle f_4, f_5, f_6 \rangle, \\ \langle f_2, f_3, f_4, f_5, f_6 \rangle \langle f_4, f_5, f_6 \rangle = 0 \end{array} \right\}$ $M_7^0 \in \mathcal{R}$ (take $f_i = e_i$ for $1 \leq i \leq 6$ ), but $g_{10}, M_1^{1,0}, M_4, g_4^C \notin \mathcal{R}$
$M_1^{1,1} \not\rightarrow M_4, g_{17}, g_{24}$	$\dim(M_4)^2 = \dim(g_{17})^2 = \dim(g_{24})^2 > \dim(M_1^{1,1})^2$
$M_3 \not\rightarrow g_5^C, g_4^C$	$\dim(g_5^C)^3 = \dim(g_4^C)^3 > \dim(M_3)^3$
$M_1^{1,0} \not\rightarrow M_1^{0,1}$	$\dim Z_2(M_1^{0,1}) < \dim Z_2(M_1^{1,0})$
$M_4 \not\rightarrow g_4^C$	$\dim(g_4^C)^3 > \dim(M_4)^3$

Table 6.3. Non-degenerations of nilpotent Malcev algebras of dimension 6.



**Figure III.** The graph of primary degenerations for six-dimensional nilpotent Malcev algebras.

**Corollary 9.**  $N\text{Mal}_6 = \{\mathcal{C}_1, \mathcal{C}_2\}$ , where  $\mathcal{C}_1 = \overline{O(g_6)} = N\text{Lie}_6$  and  $\mathcal{C}_2 = \bigcup_{\epsilon \in \mathbb{C}} \overline{O(M_6^\epsilon)} = N\text{Mal}_6 \setminus \{g_6, g_5, g_8, g_7, g_{14}, g_9, g_{23}\}$ . In particular,  $\text{Rig}(N\text{Mal}_6) = \text{Rig}(N\text{Lie}_6) = \{g_6\}$ .

**Proof.** In view of Theorem 8 it is enough to prove that  $M_6^* \not\rightarrow g_9$ ,  $M_6^* \not\rightarrow g_{23}$ ,  $M_6^* \rightarrow M_7^1$  and  $M_6^* \rightarrow M_5^1$ , where  $M_6^* = \{M_6^\epsilon\}_{\epsilon \in \mathbb{C}}$ . The first two assertions follow from the fact that  $\dim(g_9)^2 = \dim(g_{23})^2 > \dim A^2$  for any  $A \in M_6^*$ .

To prove that  $M_6^* \rightarrow M_7^1$  one can choose the parametrized basis

$$E_1^t = e_1, E_2^t = e_2 - e_4, E_3^t = te_4, E_4^t = e_3, E_5^t = e_5, E_6^t = e_6$$

and the parametrized index  $\epsilon(t) = \frac{1}{t}$ .

To prove that  $M_6^* \rightarrow M_5^1$  one can choose the parametrized basis

$$E_1^t = e_2, E_2^t = te_1, E_3^t = e_4, E_4^t = -te_3, E_5^t = -t^2e_5, E_6^t = te_6$$

and the parametrized index  $\epsilon(t) = -t^2$ .

□

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